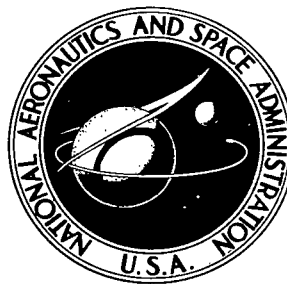


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ON THE USE OF ALGEBRAIC METHODS
IN THE ANALYSIS AND DESIGN OF
MODEL-FOLLOWING CONTROL SYSTEMS

by Heinz Erzberger

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SUMMARY

This study gives an analysis of and offers design criteria for three configurations of model following. The three configurations studied are real model following, implicit model following or matching dynamics, and real model following with command inputs. It is assumed that model and plant are described by linear vector differential equations where the equations of the model may be of lower order than those of the plant. Algebraic tests are developed to determine under what conditions a feedback law exists that permits perfect following of the model by the closed-loop plant. The same set of tests is shown to be applicable to both real and implicit model following. This leads to the conclusion that real and implicit model following, although physically different, are mathematically equivalent if no unknown state disturbances or parameter variations occur during the control interval. However, the condition for perfect following with command inputs to the model contains an additional test not present in the first two configurations. If perfect following has been shown to be possible, the control law that achieves it is calculated for the implicit model-following configuration. In the general case, this control law must generate both finite and impulse controls if the model is of lower order than the plant. A simple method of approximating the impulse control law to arbitrary accuracy is also given. The theory is illustrated with three examples, two of which are based on the lateral equation of motion of an aircraft.

INTRODUCTION

A model-following control system comprises two main components, the plant and the model. For instance, the plant is represented by the equations of motion of a particular aircraft and the model by the equations of motion of another aircraft having some especially desirable flight characteristic or handling quality. This paper is not concerned with the problem of selecting a model, but assumes at the outset that a model has already been specified. In the most general terms, the designer of a model-following control system is faced with the following problem: Given the differential equations describing the plant and the model, choose a feedback law around the plant so that the output variables of the plant, as a function of time, will faithfully follow the output variables of the model.

Recently, Ellert and Merriam (ref. 1) and Tyler (ref. 2) used quadratic optimal control to synthesize model-following control systems. Their

technique, unlike those based on classical procedures, is applicable to arbitrary multivariable systems and always yields a feedback configuration that minimizes a quadratic function of the error between the plant output and the model.

Although the application of optimal control theory to the synthesis of model-following systems was a great step forward, experience has brought to light some additional design problems for which the methods of optimal control alone prove computationally inefficient and conceptually unenlightening. An example of such a problem is that of deciding when the closed-loop plant can follow the model perfectly. That is, for a particular combination of plant and model, one may find that the closed-loop plant designed by the methods of optimal control follows the model with unacceptably large errors that cannot be reduced below some limiting value merely by manipulating the weight matrices in the cost function. Stated in another way, in a particular problem there may not be enough degrees of freedom in selecting the feedback matrix to match the plant to the model if the model dynamics differ greatly from the plant dynamics. In this case, as always, the feedback matrix calculated via optimal control still yields a weighted least-squares match between model and plant response during the control period, but gives no prior indication of matching accuracy, which must be determined separately either by actually checking the response of the closed-loop system or by evaluating the minimum cost.

Another problem that optimal control can solve only indirectly is that of deciding if the requirement of perfect matching of plant and model response entails a feedback matrix, some or all of whose elements asymptotically approach infinity. To answer this question by optimal control, one observes the elements of the optimum feedback matrix in response to a stepwise reduction of the weight on the control vector in the quadratic cost function until a conclusion about the asymptotic behavior of the elements can be drawn. Despite its obvious inefficiency, this procedure, based on repeated calculations of optimum feedback matrices, currently offers the only general approach to this question.

Finally, there is the dilemma of choosing between a design based on a "model in the system" (real model) and the so-called "model in the performance index" (implicit model). From the standpoint of mechanization, the essential difference between these two designs is that the former requires real-time comparison of model states and plant output, implying simulation of the model within the system, whereas the latter does not. But the relative merits of these two design approaches are not entirely clarified in the literature, although Tyler (ref. 2) has shed some light on this question.

Mainly, this paper concerns the development of a sequence of algebraic tests applicable to both implicit and real model following for checking whether a plant can follow a model without error. The first test of the sequence answers the question of whether there exists a finite control function for perfect following of a given plant and model. Failure of this test leads to the application of the second test in the sequence, which establishes if perfect following is possible when both finite and delta function controls are permitted, or to the termination of the testing procedure and the conclusion that no controls, neither finite nor delta functions, can achieve

perfect following. If perfect following is not achieved at the end of the second test, the same two options as at the end of the first test reoccur. That is, failure of the second test uniquely leads either to a third test enlarged to include derivatives of delta functions or to termination of the testing sequence. In the most general case, the testing may continue until $n-2$ derivatives of delta functions have been included, where n is the order of the plant. The control law that achieves perfect following of model and plant is obtained as a by-product of these tests for implicit model following.

Finally, the greater understanding of the model-following theory, exemplified, in particular, by the conditions for perfect following being identical for both implicit and real model following, leads to a rational criterion for choosing between the two design alternatives.

Conditions for perfect following are also derived for real model following with command inputs to the model. Two conditions for perfect following are obtained, the first is the same as before and the second is unique to this configuration.

The paper concludes with the discussion of three examples. The first, based on the lateral equations of motion of an aircraft, compares the performance of implicit and real model-following designs under various operating conditions. The second and third examples were chosen primarily to illustrate the theory when unbounded controls are required for perfect matching.

SYMBOLS

Capital letters such as F , B , H denote matrices whose dimensions are defined in the text. The exceptions to this rule are

I unity matrix

J loss function

K gain constant defined in the text

Small letters denote state, output, or control variables.

A' transpose of A

A^\dagger pseudoinverse of A (also known as generalized inverse)

L^{-1} inverse Laplace transform

$\mathfrak{N}(A)$ null space of A

$\mathfrak{N}(A)^\perp$ perpendicular complement of $\mathfrak{N}(A)$

$\mathcal{R}(A)$ range of matrix A

$R(A)^\perp$	perpendicular complement of $R(A)$
\dot{y}	identical with $\frac{dy}{dt}$
ϵ	either a small number or the logical statement "is a member of," depending on context
$\delta(t-\tau)$	delta function at $t = \tau$
$\delta^j(t-\tau)$	(j-1)th derivative of delta function
$\Psi \subset \Omega$	set Ψ is contained in set Ω
$\Psi \cap \Omega$	logical intersection of sets Ψ and Ω

IMPLICIT MODEL FOLLOWING

In implicit model following, the output dynamics of the plant are modified by means of feedback to approximate the dynamics of a given model. Instead of minimizing the error between plant output and model states directly, implicit model following imposes a somewhat weaker condition which is stated in mathematical terms as follows. Let the multivariable plant be described by

$$\dot{x} = Fx + Bu \quad (1)$$

$$y = Hx \quad (2)$$

where x is an n -dimensional state vector, u an m -dimensional control vector, and y an l -dimensional output vector. The matrices F , B , and H do not depend on time and have dimensions $n \times n$, $n \times m$, and $l \times n$, respectively. Also, it is assumed that $n \geq m$ and $n \geq l$. The mathematical description of the model is taken to be

$$\left. \begin{aligned} \dot{z} &= Lz \\ L &= l \times l \quad \text{constant matrix} \end{aligned} \right\} \quad (3)$$

where z denotes the l -dimensional state vector of the model. The objective of implicit model following is to find a feedback law $u = Sx$ to be placed around the plant so that the output y approximates

$$\dot{y} = Ly \quad (4)$$

as closely as possible over some specified time interval. One technique for achieving this objective is to use optimal control to calculate the control law that minimizes the following quadratic loss function (refs. 3 and 4):

$$J = \int_0^T [(\dot{y} - Ly)'Q(\dot{y} - Ly) + u'Ru]dt$$

where Q is a positive semidefinite and R , a positive definite matrix. This formulation of model following does not introduce the state variables of the model directly, since y and \dot{y} appearing in the loss function can be expressed by means of equations (1) and (2) in terms of x and u alone. Hence, the terms "implicit model," "model in the performance index" (ref. 2), and "matching dynamics" (ref. 5) have all been used to describe this method. Figure 1 illustrates the absence of dynamic coupling between model states and the closed-loop plant characteristic of this configuration.

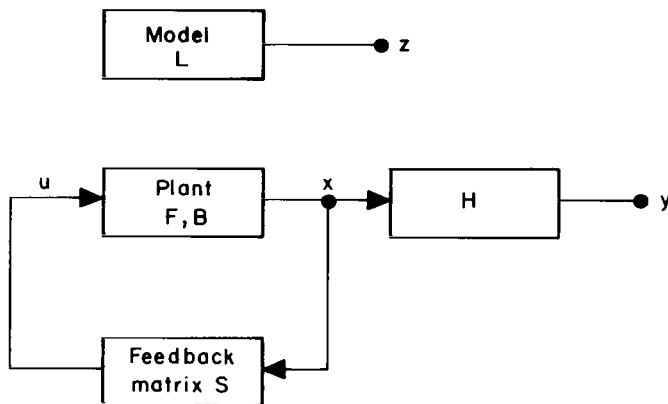


Figure 1.- Implicit model following.

We now address ourselves to the main problem of this paper, namely, that of determining under what conditions plant and model can be matched exactly. In essence, this problem is equivalent to determining whether or not the elements of the feedback matrix S offer the designer enough freedom of choice so that he can force the output of the closed-loop plant to satisfy equation (4) exactly. This question suggests that algebraic methods might provide a convenient framework for its solution. Using equations (1) and (2) and

requiring that equation (4) be a strict equality permits us to write

$$HBu = (LH - HF)x \quad (5)$$

If a control u corresponding to any x in the state space is to exist so that equation (5) is satisfied, then the range of HB must contain the range of $(LH - HF)$:

$$\mathcal{R}(HB) \supset \mathcal{R}(LH - HF) \quad (6)$$

To derive an algebraic condition equivalent to condition (6), equation (5) is formally solved for u by taking the pseudoinverse of HB (ref. 6):

$$u = (HB)^\dagger (LH - HF)x \quad (7)$$

Then, after u is eliminated from equations (5) and (7), the condition for perfect matching becomes

$$[(HB)(HB)^\dagger - I](LH - HF)x = 0 \quad \text{all } x \quad (8)$$

To justify the use of the pseudoinverse, one must show that if equation (8) is true for all x , that is, $[(HB)(HB)^\dagger - I](LH - HF)$ is the zero transformation, then condition (6) is a necessary consequence. A property of the pseudoinverse

that permits this conclusion is the fact that $(HB)(HB)^\dagger$ is an orthogonal projection operator on $\mathcal{R}(HB)$. For the proof, let z be any vector in $\mathcal{R}(LH - HF)$ and write it as the sum

$$z = z_0 + z_\perp$$

where

$$z_0 \in \mathcal{R}(HB) \quad \text{and} \quad z_\perp \in \mathcal{R}(HB)^\perp$$

Since $[(HB)(HB)^\dagger - I]$ is also an orthogonal projection that projects every $z \in \mathcal{R}(LH - HF)$ on $\mathcal{R}(HB)^\perp$ and since, by assumption, equation (8) is zero for every x , it follows that

$$z = z_0$$

and, therefore, $\mathcal{R}(LH - HF) \subset \mathcal{R}(HB)$. Thus, we conclude that choosing

$$u = (HB)^\dagger(LH - HF)x \tag{9}$$

when

$$[(HB)(HB)^\dagger - I](LH - HF) = 0 \tag{10}$$

guarantees that $\dot{y} = Ly$, or, equivalently, that the output dynamics of the closed-loop system will match the desired output dynamics. Furthermore, the boundedness of the pseudoinverse implies that the feedback law $(HB)^\dagger(LH - HF)$ is bounded. Therefore, if the condition for zero error (eq. (10)) is satisfied and the model is stable, the controls that achieve a perfect match are always bounded.

This proves the sufficiency of equation (10) for perfect matching with bounded controls. The necessity of this equation follows from a similar argument. For if equation (10) is not the zero transformation, then any x such that

$$[(HB)(HB)^\dagger - I](LH - HF)x \neq 0$$

must necessarily yield a $z = z_0 + z_\perp$ where $z_\perp \neq 0$. Since this z_\perp is not within the range of HB , it follows that no u exists which can satisfy equation (5) for that particular choice of x .

When equation (10) is not satisfied, it may still be possible to achieve zero error by enlarging the class of controls to include delta functions. As the next step, the control law and test for perfect matching derived above is extended to the case of unbounded controls. One begins by writing every control as a direct sum of ordinary and delta (impulse) functions:

$$u(t, \tau) = u_1(t) + u_8(\tau)\delta(t - \tau) \quad (11)$$

$$u_1 \in \mathcal{N}(\text{HB})^\perp$$

$$u_8 \in \mathcal{N}(\text{HB})$$

where \mathcal{N} denotes null space, \perp the perpendicular complement, τ a running variable, and t current time assumed to be fixed. Thus, the delta function occurs at time t . The strength of the delta function, u_8 , is restricted to lie in the null space of HB ; otherwise the left side of equation (5) would contain an impulse of strength $\text{HB}u_8$ while the right side would not. Hence, perfect matching would be absent at the moment the impulse occurred.

It is shown in appendix A that at time t^+ , a moment after the impulse has occurred, the derivative of the output $\dot{y}(t^+)$ is as follows:

$$\dot{y}(t^+) = \text{HF}x + \text{HB}u_1(t) + \text{HFB}u_8(t) \quad (12)$$

Our objective is to make the right side of equation (12) equal to $\text{LH}x(t)$ by appropriately choosing $u_1(t)$ and $u_8(t)$:

$$\text{HB}u_1(t) + \text{HFB}u_8(t) = (\text{LH} - \text{HF})x(t) \quad (13)$$

Before equation (13) can be solved explicitly for the control functions, it is necessary to introduce an auxiliary variable $\tilde{u} \equiv u_1 + u_8$ and write u_1 and u_8 , by means of the pseudoinverse, as projections of \tilde{u} on $\mathcal{N}(\text{HB})^\perp$ and $\mathcal{N}(\text{HB})$, respectively:

$$u_1 = (\text{HB})^\dagger (\text{HB})\tilde{u}, \quad u_8 = [\text{I} - (\text{HB})^\dagger \text{HB}]\tilde{u} \quad (14)$$

Then, after substituting equations (14) into equation (13), one may solve explicitly for \tilde{u} :

$$\tilde{u} = \text{M}^\dagger (\text{LH} - \text{HF})x \quad (15)$$

where

$$\text{M} = \text{HB} + \text{HFB}[\text{I} - (\text{HB})^\dagger \text{HB}]$$

Finally, the condition for perfect matching can now be derived by replacing u_1 and u_8 in equation (13) with the relationship for these quantities obtained from equations (14) and (15):

$$(\text{MM}^\dagger - \text{I})(\text{LH} - \text{HF}) = 0 \quad (16)$$

If condition (16) is satisfied, then equation (15) essentially gives the control law that achieves perfect matching, except that the problem of generating a delta function control has not yet been considered. Assuming for the moment that it is possible to generate the required delta function, we want to demonstrate that from time t^+ onward the equality of equation (4) can be maintained. In general, equation (4) or (5) will not be equal at time t since the effect of the delta function is not felt until time t^+ , an

infinitesimal instant later than t . At that moment a step change occurs in \dot{y} in such a way that equation (4) becomes an equality. Perfect matching is therefore assured for at least a time interval that is short in comparison to the fastest time constants of the system. As soon as the difference between \dot{y} and Ly exceeds some small threshold, where the value of the threshold may be chosen arbitrarily small, another impulse whose weight is chosen according to equations (14) and (15) is applied. The second impulse reestablishes the equality of equation (4). Clearly, perfect matching can thus be maintained indefinitely by applying impulses whenever the threshold value is exceeded. We also note that the smaller the threshold value, the closer will be the spacing of the impulses, but also the smaller will be the strength of the impulses.

The problem of implementing a closed-loop control law that generates the required delta functions is discussed in appendix B where it is shown that an approximate synthesis of such a control law is obtained by multiplying u_δ by a large positive gain constant K and that the approximation to the ideal delta function control law improves in proportion to the magnitude of K .

If equation (16) is not satisfied and the rank of M is not maximal, it may still be possible to achieve perfect matching by including derivatives of delta functions in addition to the previously used controls. The derivation in this case will be indicated only briefly since the arguments remain essentially the same as before. The control u is now written as the sum of three controls:

$$u = u_1 + u_\delta \delta(t - \tau) + u_{\delta 1}(t - \tau)$$

$$u_1 \in [\mathcal{N}(\text{HB})]^\perp$$

$$u_\delta \in \mathcal{N}(\text{HB}) \cap [\mathcal{N}(\text{HFB})]^\perp$$

$$u_{\delta 1} \in \mathcal{N}(\text{HB}) \cap \mathcal{N}(\text{HFB})$$

This decomposition insures that delta functions and their derivatives do not appear in the expression for \dot{y} , which becomes, according to appendix A,

$$\dot{y} = \text{HF}x + \text{HB}u_1 + \text{HFB}u_\delta + \text{HF}^2\text{B}u_{\delta 1} \quad (17)$$

At this point, the right side of \dot{y} is equated to $\text{LH}x$ and the condition is found that describes when the resulting equation has a solution u for all x in the state space. Fundamentally, the procedure for deriving this condition is similar to the previously treated case of finite and delta function controls. One begins by defining $\tilde{u} = u_1 + u_\delta + u_{\delta 1}$ and then writing the three components of \tilde{u} as orthogonal projections of \tilde{u} on the appropriate subspaces:

$$u_1 = P_1\tilde{u}, \quad u_\delta = P_\delta\tilde{u}, \quad u_{\delta 1} = P_{\delta 1}\tilde{u} \quad (18)$$

whereupon equations (18) are used to write equation (17) in terms of \tilde{u} alone. From that point the derivation is exactly the same as before, and the condition for perfect matching can again be put in the form given by equation (16), except that M must be taken as $M = \text{HB} + \text{HFB}P_\delta + \text{HF}^2\text{B}P_{\delta 1}$. For general

vector controls, the construction of the projections may be computationally somewhat lengthier than before, but for scalar controls explicit formulas can still be given:

$$P_1 = (HB)^\dagger HB, \quad P_8 = [I - (HB)^\dagger HB], \quad P_{81} = [I - (HB)^\dagger HB][I - (HFB)^\dagger HFB]$$

It may be verified that for this special case of scalar controls, at most one of the three projections is nonzero. If all three are zero, one may proceed to higher order delta functions by obvious extensions of the theory. In the general case of vector controls, the end of the testing sequence is reached either when the $(n-2)$ th derivative of delta functions is included or when the intersection of the null spaces $\mathcal{N}(HB)$, $\mathcal{N}(HFB)$, . . . , $\mathcal{N}(HF^jB)$ becomes empty for some $j \leq n-2$.

REAL MODEL FOLLOWING

In the implicit model-following design, only the model parameters were considered in the selection of the feedback matrix; the model states themselves were not needed. In real model following, the model states must be generated because here we ask for a feedback law S and a feedforward law A so that the response of the closed-loop system $\dot{x} = Fx + BSx + BAz$ is such that $y(t) \approx z(t)$ rather than $\dot{y} \approx Ly$ as for implicit model following. Figure 2 with $u_c = 0$ is a block diagram for this model-following configuration.

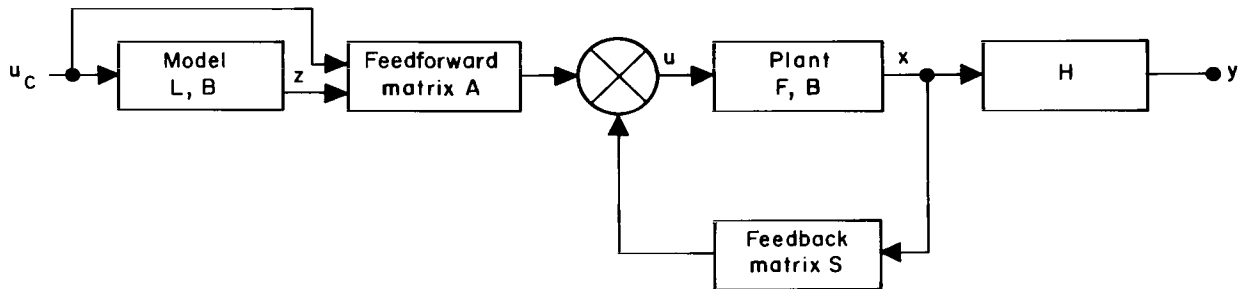


Figure 2.- Real model following.

For general multivariable systems, optimal control theory again offers the most efficient method of calculating the appropriate feedback and feedforward matrices. A convenient choice for the loss function is

$$J = \int_0^T [(y - z)'Q(y - z) + u'Ru]dt \quad (19)$$

It is customary to reduce equation (19) to standard form by augmenting the state equations of the plant with those of the model, thus forming the augmented state equation:

$$\dot{w} = \begin{bmatrix} F & 0 \\ 0 & L \end{bmatrix} w$$

where

$$w = \begin{bmatrix} x \\ \text{---} \\ z \end{bmatrix}$$

Then one minimizes the completely equivalent loss function

$$J = \int_0^T (w' \hat{Q} w + u' R u) dt \quad (20)$$

where

$$\hat{Q} = \begin{bmatrix} H' Q H & -H' Q \\ -Q H & Q \end{bmatrix}$$

It has been shown that the feedback matrix which minimizes J of equation (19) depends only on F , Q , and R and not on the model parameter L , whereas the optimum feedforward matrix depends on both model and plant parameters (ref. 2).

On first sight, it might appear that having the model in the system and using the states of the model in real time to control the plant through the feedforward loop would greatly add to the power of the method in comparison to implicit model following. Yet the two methods are equivalent in a certain sense, as will now be demonstrated.

After this brief introduction to real model following, we now proceed with the problem of deriving conditions for perfect following of plant and model when the criterion is equality of $y(t)$ and $z(t)$ for all t . The derivation uses the fact that if all orders of time derivatives of y and z are equal at time $t = 0$, then the error will be zero for all time. Beginning with the zeroth derivative, one obtains

$$z(t) = Hx(t) \quad (21)$$

at $t = 0$. This is merely an initial alinement condition of model and plant states that can always be satisfied at the start of the control period. Next, both sides of equation (21) are differentiated and then rewritten with the help of equations (1), (2), (3), and (21) as

$$H(Fx + Bu + BAz) = H(Fx + Bu + BAHx) = LHx \quad (22)$$

Here Az represents the feedforward control, but the explicit dependency of this control on the model states z has been eliminated in the middle member of equation (22) by use of equation (21). Solving the last two members of equation (22) for u , one obtains

$$u = (HB)^{\dagger}(LH - HF - HBAH)x \quad (23)$$

The condition for equality of the first derivative is found by eliminating u between equation (23) and the last two members of equation (22):

$$[(HB)(HB)^{\dagger} - I](LH - HF) = 0 \quad (24)$$

where the two terms arising from the feedforward control canceled by virtue of the identity $(HB)(HB)^{\dagger}(HB) = HB$ (ref. 6). But equation (24), if true, holds for arbitrary t ; therefore, all higher order derivatives of the error will also be zero. If condition (24) is not satisfied and HB does not have maximal rank, we can attempt to achieve perfect following by enlarging the class of admissible controls to include various orders of delta functions. But the procedure for doing this differs in no way from that for implicit model following and, moreover, yields identical results. Thus, the conditions for perfect dynamic matching are in every respect the same as for implicit model following. Direct use of the model states in the feedforward loop neither adds to nor subtracts from these conditions.

Real Model Following With Command Inputs

In some applications, the model receives command inputs from an external source, such as a human operator. This is called "flying the model" and is illustrated in figure 2. It is easy to extend the theory derived in the preceding section to such a situation. The equation for the model is

$$\dot{z} = Lz + \hat{B}u_c \quad (25)$$

where u_c is the command input. The plant, the output equation, and the objective, equation (21), remain as in real model following. Then, for perfect dynamic matching the alinement condition must be satisfied as before:

$$Hx(t) = z(t)$$

The control law for dynamic matching is obtained by requiring that the first derivative of y and z be equal:

$$u(t) = (HB)^{\dagger}(LH - HF)x(t) + (HB)^{\dagger}\hat{B}u_c(t) \quad (26)$$

The feedforward control Az has been omitted from equation (26) since it was previously shown to be irrelevant. The first derivatives of y and z will be equal for all x and u_c if

$$[(HB)(HB)^{\dagger} - I](LH - HF)x = [(HB)(HB)^{\dagger} - I]\hat{B}u_c \quad (27)$$

Since u_c is independent of x , each side of equation (27) must be zero separately:

$$\left. \begin{aligned} [(HB)(HB)^{\dagger} - I](LH - HF) &= 0 \\ [(HB)(HB)^{\dagger} - I]\hat{B} &= 0 \end{aligned} \right\} \quad (28)$$

Equations (28) give the conditions for dynamic matching with command inputs to the model. Here the second of equations (28) introduces a genuinely new condition. Again the theory can be extended as needed to include delta function controls.

EVALUATION OF REAL AND IMPLICIT MODEL FOLLOWING

It has been shown that the conditions for perfect matching and the control law which achieves perfect matching are identical for both real and implicit model following. Thus, if it is assumed that perfect matching is possible with either bounded or unbounded controls and that unknown disturbances are absent, there is no essential advantage of one design over the other. The key issue in deciding between a real model-following design (with its additional hardware requirements) and the simpler implicit model following is whether or not the requirements of the problem dictate that a particular phase trajectory of the model be followed in the presence of unknown disturbances in the plant. Implicit model following is not capable of following a phase trajectory of the model where disturbances are present since no real-time error measurement between model and plant states takes place; the model following is open loop as it were. But, if the model serves merely to characterize the desired dynamic properties of the plant, in other words, if model and plant have similar responses when starting at the same initial states with no disturbances present, the implicit model following would be sufficient.

Maintaining alinement between plant and model in the presence of uncertainties, be they unknown parameters or random disturbances, necessitates the use of a real model in the system. With a model in the system, errors arising between model and plant states due to uncertainties can be measured and corrected continuously. Thus, the principal advantage of having a model in the system is not that it always achieves better following, but that it desensitizes the following to unknown disturbances.

For real model following, the control law given by equation (23) cannot be used by itself since it does not include the states of the model; that is, this control provides dynamic matching only and does not attempt to realine the plant and the model states if disturbances cause them to drift apart. Here the techniques of optimal control would seem most appropriate for computing the control law.

EXAMPLES

In this section three examples are presented. The first represents the linearized lateral equations of motion of an aircraft. Three model-following designs, one calculated by the theory developed in this paper and the other two by the methods of quadratic optimal control, are compared, and the advantages of each are pointed out. The second and third examples illustrate the theory when unbounded control laws are required for perfect matching. All computations were performed with the automatic synthesis program of Kalman and Englar (ref. 5).

Example 1

The numerical values for both plant and model parameters were taken from one of Tyler's examples (ref. 2). The plant actually describes the lateral dynamics of the B-26 airplane which was used as a test bed for model-following studies at Cornell Aeronautical Laboratory, whereas the model corresponds to the lateral dynamics of some other airplane whose handling characteristics are to be simulated on the B-26.

$$F = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -2.93 & -4.75 & -0.78 \\ 0.086 & 0 & -0.11 & -1.0 \\ 0 & -0.042 & 2.59 & -0.39 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & -73.14 & 3.18 \\ 0.086 & 0 & -0.11 & -1 \\ 0.0086 & 0.086 & 8.95 & -0.49 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix}$$

$$\text{State vector} = \begin{bmatrix} \phi \\ \dot{\phi} \\ \beta \\ r \end{bmatrix} \begin{array}{l} \text{(bank angle)} \\ \text{(bank rate)} \\ \text{(sideslip angle)} \\ \text{(yaw rate)} \end{array}$$

$$\text{Control vector} = \begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} \begin{array}{l} \text{(rudder deflection)} \\ \text{(aileron deflection)} \end{array}$$

$$H = I$$

The test of perfect matching (eq. (8)) applied to this example gives

$$[(HB)(HB)^{\dagger} - I](LH - HF) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1.3 \times 10^{-7} & -4.3 \times 10^{-6} & -1.1 \times 10^{-5} & -3.4 \times 10^{-6} \\ -1.2 \times 10^{-4} & -3.9 \times 10^{-3} & -1.3 \times 10^{-2} & -3.0 \times 10^{-3} \\ -1.6 \times 10^{-6} & -5.4 \times 10^{-5} & -1.8 \times 10^{-4} & -4.0 \times 10^{-5} \end{bmatrix} \quad (29)$$

Thus, perfect matching is not possible because the right side of equation (29) is not the zero transformation. Since HB has maximum rank, it also follows from earlier work that delta function controls cannot improve this situation. Nevertheless, because most entries in the matrix of equation (29) are small compared with entries in the system and model matrices, it is interesting to compare the performance of the simple model-following control law, equation (7),

$$u = (HB)^{\dagger}(LH - HF)x = \begin{bmatrix} -3.4 \times 10^{-3} & -0.11 & -0.37 & -0.084 \\ 0 & -0.49 & 17.5 & -1.01 \end{bmatrix} x \quad (30)$$

with those calculated by optimal control for both the implicit and real model-following performance indices. For the implicit model-following case the diagonal weight matrices Q and R selected for this calculation correspond to those used by Tyler and appear to give a reasonable compromise in matching all the state variables. The Q and R matrices entering into the real model-following performance index were selected experimentally to give good following of all model states. The numerical values for the Q, R, feedback, and feedforward matrices corresponding to the two optimal control designs are given below.

Implicit model following

$$\text{Diagonal } Q = [0, 6, 0, 6] \quad \text{Diagonal } R = [1, 1]$$

$$\text{Feedback matrix} = - \begin{bmatrix} 0.0034 & 0.111 & 0.371 & 0.0356 \\ 0 & 0.494 & -17.5 & 0.614 \end{bmatrix}$$

Real model following

$$\text{Diagonal } Q = [10, 10, 10, 10] \quad \text{Diagonal } R = [1, 1]$$

$$\text{Feedback matrix} = \begin{bmatrix} -0.074 & -0.094 & 2.34 & -3.23 \\ -3.15 & -2.73 & 0.835 & 0.261 \end{bmatrix}$$

$$\text{Feedforward matrix} = \begin{bmatrix} 0.031 & -0.246 & 2.109 & -4.62 \\ -3.02 & -3.16 & 16.6 & -2.3 \end{bmatrix}$$

Figures 3 and 4 compare the transient responses of the three different control laws for two initial conditions corresponding to an initial bank angle and an initial bank rate, but with the model and plant states aligned at the start. Because the response of the implicit model-following law calculated with optimal control was generally not much different from the response obtained with the control law of equation (30), it is not drawn on all the figures in order to reduce crowding of the curves. Also, those state variable time histories not included were found to be as well matched as ϕ in figure 3(a). It can be seen in figures 3 and 4 that at least during the first 5 seconds, the control law of equation (30) performs as well as the real model-following design or better. This is particularly evident in the responses of figures 3(b), 3(c), and 4(b), where the real model-following design often shows considerable error between model and plant. Experimentation with the Q matrix did not measurably improve the matching. A probable explanation of why the control law of equation (30) achieves better matching of the transient response is suggested by the fact that optimal control minimizes the square of the error over the entire control interval, which in

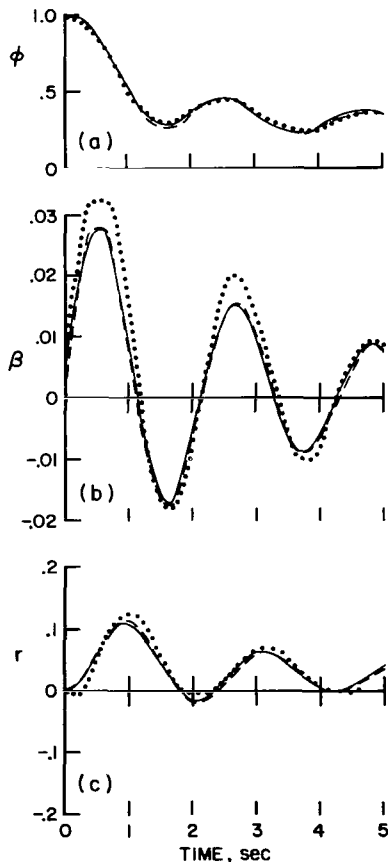


Figure 3.- Transient responses pertaining to example 1: initial bank angle disturbances.

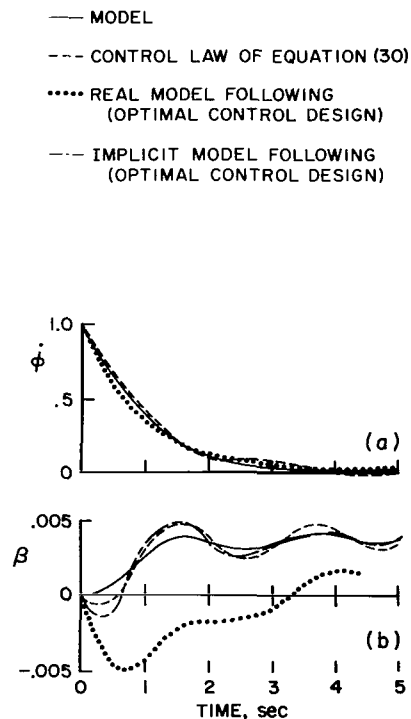


Figure 4.- Transient responses pertaining to example 1: initial roll rate disturbance.

this case was essentially infinite. This would put primary emphasis on the steady-state portion of the response and would deemphasize matching of the transient response. The opposite is true of control law (30) which essentially tries to minimize the norm of the error between time derivatives of model and plant states at any given moment. This is due to a property of the pseudoinverse (ref. 6). When model and plant are not too dissimilar, as shown in this case by the results of the perfect following test (eq. (29)), one can expect this procedure to work quite well; but for a greatly mismatched model and plant, again as determined by the perfect following test, no assurance of satisfactory operation can be given. This line of reasoning also explains why the response of control law (30) approximates the implicit model-following response obtained by optimal control, because both attempt to match the derivatives of model and system states, although the latter does so over an infinite time interval.

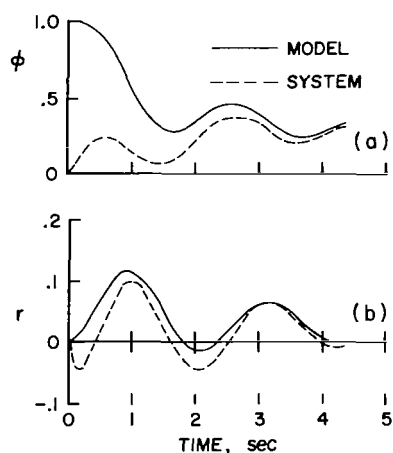


Figure 5.- The effect of initial bank misalignment between model and plant in real model following with optimal control design.

Slight discrepancies between the transient responses of the real model-following design calculated by Tyler (ref. 2) and those calculated in this paper have been observed. The origin of the discrepancies has not been definitely established, but it probably lies in small differences between the parameters used in the two studies.

Figures 5(a) and 5(b) demonstrate when it is advantageous to use real model following. Here a disturbance in the plant is assumed to have caused a sudden misalignment between the model and plant bank-angle variables. Under this condition, the real-time error measurement between model and plant, which is possible only with real model following, facilitates the eventual realignment of

corresponding state variables. Thus, the model serves as a memory of a particular trajectory in the presence of disturbances.

Example 2

A simple example is chosen deliberately so that the required computation can be performed by hand. The plant, output, and model parameters are

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$H = [1 \quad 0] \quad L = -5$$

Since $HB = 0$, we conclude that finite controls will not be sufficient to achieve perfect matching. But the perfect matching test based on the use of delta function controls, equation (16), can be successful. Then, according to appendix B, the control law that achieves perfect matching asymptotically as K approaches infinity is

$$u = K \begin{bmatrix} -5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A block diagram of the closed-loop system is shown in figure 6.

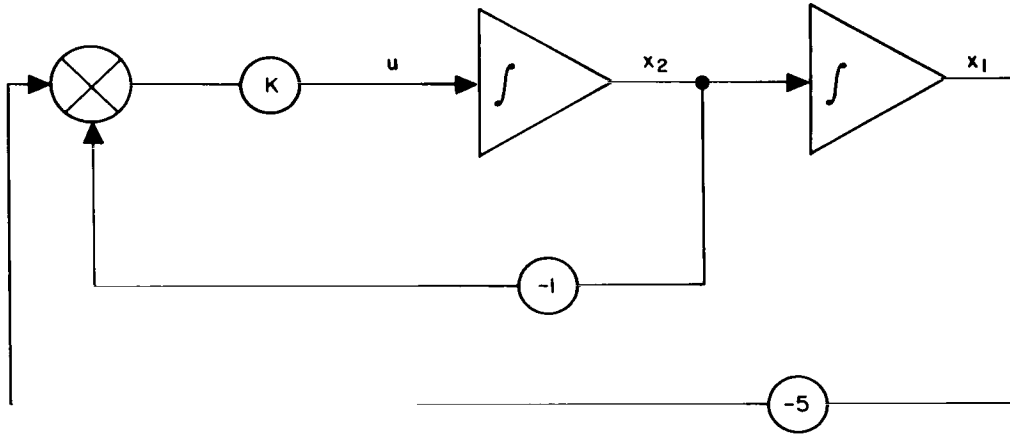


Figure 6.- Closed loop system of example 2.

Example 3

The open-loop plant equations of this example are fourth order and differ only slightly from those of example 1. The main difference lies in the model which is only second order:

$$F = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -2.93 & -4.75 & 0 \\ 0.085 & 0 & -0.11 & -1.0 \\ 0 & -0.042 & 2.59 & -0.39 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

Thus, the objective in this case is to make the essentially fourth-order plant look like a simple second-order system through the appropriate use of feedback.

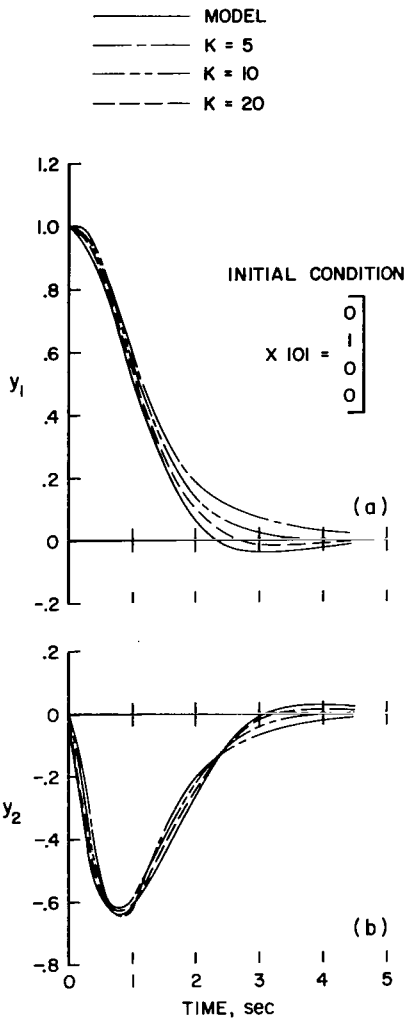


Figure 7.- The effect of K on the transient response of example 3.

It can be shown that the test for perfect following with finite controls (eq. (10)) applied to this example fails; therefore it is necessary to use the more general test given by equation (16), which considers sums of finite and delta function controls. This latter test shows that perfect following is indeed possible with a control law containing both finite and delta function controls.

After the required calculations are performed, the two parts of the control law are

$$u_1 = \begin{bmatrix} 0 & 1.465 & 2.875 & 0 \\ 0 & 1.465 & 2.875 & 0 \end{bmatrix} x$$

$$u_8 = \begin{bmatrix} -0.086 & -2 & -1.89 & 1 \\ 0.086 & 2 & 1.89 & -1 \end{bmatrix} x$$

As shown in appendix B, an approximate synthesis of a control law containing delta functions is obtained by multiplying u_8 by a large positive constant K . The total control is then given by the sum of the two components, with K appearing as a parameter in the feedback matrix:

$$u = \begin{bmatrix} -0.086K & 1.465-2K & 2.875-1.89K & K \\ 0.086K & 1.465+2K & 2.875+1.89K & -K \end{bmatrix} x$$

Figures 7(a) and 7(b) demonstrate the convergence properties of the control law as a function of the gain constant K .

CONCLUSIONS

Two basic approaches, each having its particular advantages and disadvantages, exist for designing a model-following control system. In the implicit model-following method, the model enters only into the selection of the feedback law placed around the plant but does not become physically part of the total system. Thus, feedback is used to modify the dynamics of the plant so that its output behavior coincides with that of the model. This type of following therefore operates open loop with respect to the model since during the control interval no real-time comparison of model states and plant output

takes place. The main advantages of this method are simplicity and low cost of implementation because simulation of the model in the system is not necessary.

If design specifications require that the model-following control system be able to follow a specific phase trajectory of the model starting at a given initial state while the plant is subject to unknown disturbances or parameter changes, then real model following is the appropriate choice. Here the continuous measurement of error between model states and plant output offers the additional freedom of using this error, appropriately weighted, as a means of aligning the model and the plant. However, because the conditions for perfect following are identical for both real and implicit model following, this additional freedom does not contribute to improved matching of the dynamics of model and plant in comparison with implicit model following.

Although optimal control theory offers the most general method available for the design of model-following systems, it is inefficient, because of the computational effort required, for answering such preliminary design questions as whether or not it is possible to match model and plant dynamics and, if so, whether bounded or unbounded controls are required. The theory presented here answers such questions directly by means of an algebraic test and, in addition, furnishes, for implicit model following, a simply computed control law that achieves perfect matching if known to be possible by the test. It is demonstrated that even if perfect matching is not possible the performance of a system using the simple control law may compare favorably with the performance of systems designed via optimal control so long as the dynamics of model and plant are not too dissimilar.

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APPENDIX A

CALCULATION OF OUTPUT DERIVATIVE FOR INPUTS CONTAINING DELTA FUNCTIONS

The purpose of this appendix is to derive expressions for the derivative of the output, $\dot{y}(t)$, when the controls applied to the system contain various orders of delta functions. For reasons discussed in the main body of this report, not every possible combination of finite and delta function controls need be considered here; rather, it is sufficient to let the control $u(t)$ be of the form

$$u(t, \tau) = u_1(t) + u_\delta \delta(t - \tau) + u_{\delta^1} \delta(t - \tau) + \dots + u_{\delta^{n-2}} \delta^{n-2}(t - \tau) \quad (A1)$$

where

$$\begin{aligned} u_1 &\in \mathfrak{M}(\text{HB})^\perp \\ u_\delta &\in \mathfrak{M}(\text{HB}) \cap \mathfrak{M}(\text{HFB})^\perp \\ u_{\delta^1} &\in \mathfrak{M}(\text{HB}) \cap \mathfrak{M}(\text{HFB}) \cap \mathfrak{M}(\text{HF}^2\text{B})^\perp \\ &\vdots \\ u_{\delta^{n-2}} &\in \mathfrak{M}(\text{HB}) \cap \dots \cap \mathfrak{M}(\text{HF}^{n-2}\text{B}) \cap \mathfrak{M}(\text{HF}^{n-1}\text{B})^\perp \end{aligned}$$

If, for the moment, one considers an arbitrary delta function $\hat{u}_\delta(t, \tau) = \delta(t - \tau)u_\delta(\tau)$ and a bounded control $u_1(t)$, then $\dot{y}(t)$ is written

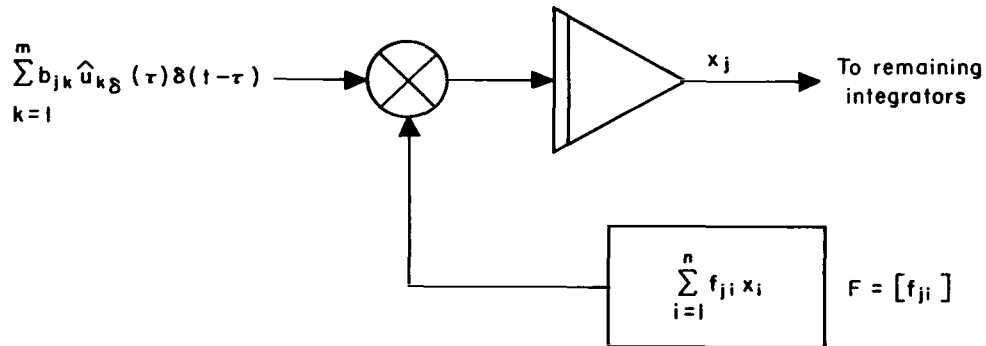
$$\dot{y}(t) = H[Fx(t) + Bu_1(t) + B\hat{u}_\delta(t, \tau)\delta(t - \tau)] \quad (A2)$$

We are interested in calculating $\dot{y}(t^+)$ where t^+ designates, as is customary, a time constant infinitesimally later than t . It will now be shown that $\dot{y}(t^+)$ has the form

$$\dot{y}(t^+) = H\{F[x(t) + B\hat{u}_\delta(t)] + Bu_1(t)\} \quad (A3)$$

assuming $u_1(t)$ and $u_\delta(t)$ are continuous functions of t . To prove this fact, sketch (a) assumes that the j th state variable x_j receives $\hat{u}_\delta(\tau)\delta(t - \tau)$ through the distribution matrix B , whose components are designated by b_{ik} . The action of the delta function causes the output of the j th integrator at time t^+ to take the value $\hat{x}_j(t^+) = x_j(t) + \sum b_{jk}\hat{u}_k(t)$. Similar reasoning must be applied to the output of the remaining integrators, thus yielding $\hat{x} = x + Bu_\delta$. This updated state vector \hat{x} was used in deriving equation (A3). Finally, the condition that $\hat{u}(\tau) = u_\delta(\tau)\mathfrak{M}(\text{HB})$ reduces equation (A2) to

$$\dot{y}(t) = H\dot{F}x(t) + HBu_1(t) \quad (A4)$$



Sketch (a).

that is, the impulse at time t causes a step change in the derivative of y , but the impulse itself has been prevented from appearing in \dot{y} .

If y is of the general form given by equation (A1), it is easy to demonstrate that equation (A3) generalizes to

$$\dot{y}(t^+) = H[Fx(t) + Bu_1(t) + FBu_{\delta}(t) + \dots + F^{n-1}Bu_{\delta n-2}(t)] \quad (A5)$$

APPENDIX B

APPROXIMATE SYNTHESIS OF FEEDBACK LAW CONTAINING DELTA FUNCTIONS

In the derivation of conditions for perfect following, it was necessary to include delta functions as permissible controls in order to achieve perfect following. Whenever the model is of lower order than the plant, it may be expeditious to sacrifice a part of the plant dynamics for better matching, and in that case delta function controls are necessary. Important questions arise now as to the procedure for constructing a control law containing delta functions and how to approximate one to arbitrary accuracy. Although the concept behind the construction of such a control law is well known, its adaptation to this problem requires some explanation. The procedure involved will be illustrated on a controllable plant with scalar control.

Assume now that the perfect matching test developed earlier has been performed and that the required control law for perfect matching is

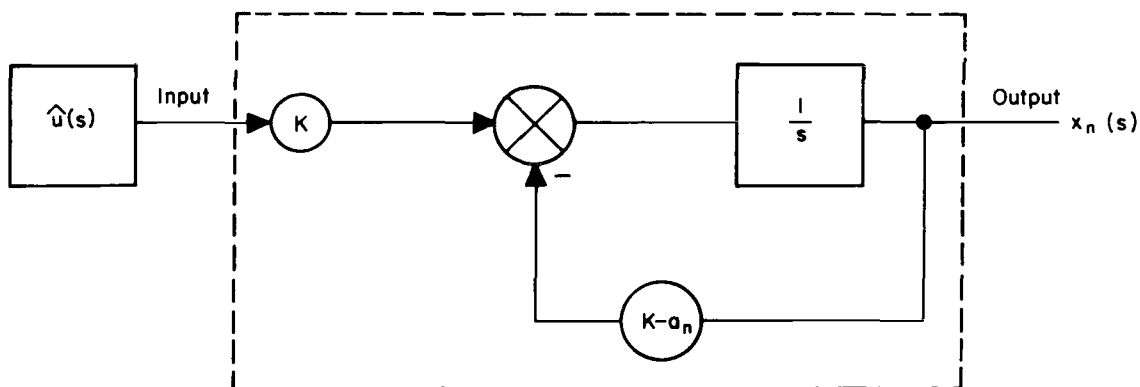
$$u = (\text{HFB})^\dagger (\text{LH} - \text{HF}) \delta(t - \tau) x \quad (\text{B1})$$

Equation (11) always reduces to equation (B1) if u is a scalar. The principal conclusion of this appendix is that the control law given by equation (B1) can be approximated to arbitrary accuracy in a given time interval $0 < t \leq T$ by

$$u = K(\text{HFB})^\dagger (\text{LH} - \text{HF}) x \quad (\text{B2})$$

where the positive constant K is chosen sufficiently large.

For the proof, let $\hat{u}(s)$ be a rational function of s with the denominator at least one degree higher than the numerator, and consider the output $x_n(t)$ of the system shown in sketch (b). Then one can show that the output



Sketch (b).- Canonical system for generating delta function controls.

$x_n(t)$ of this system can be expressed as

$$x_n(t) = L^{-1}\hat{u}(s) + x_n(0)e^{-(K-a_n)t} + Ku(-K + a_n)e^{-(K-a_n)t} + \sum_{j=1}^p \frac{R_j(s_j - a_n)}{s_j + K - a_n} e^{ts_j} \quad (B3)$$

where R_j is the residue of the j th pole of $\hat{u}(s)$.

A word of explanation on deriving equation (B3) is in order. One starts with $x_n(t)$ obtained by standard transform methods:

$$x_n(t) = L^{-1}\left[\frac{x_n(0)}{s + K - a_n}\right] + L^{-1}\left[\frac{\hat{u}(s)K}{s + K - a_n}\right]$$

Then follows the crucial step of adding and subtracting $L^{-1}\hat{u}(s)$ to the right-hand side of the equation, whereupon routine algebraic manipulations yield the form of $x_n(t)$ given in equation (B3). The purpose of writing $x_n(t)$ in this way is to find an explicit expression for the error as a function of K between the desired time response $L^{-1}u(s)$ and the actual time response, $x_n(t)$, with the ultimate objective of showing that this error approaches zero as K becomes infinite. With respect to equation (B3), this error, denoted by ϵ and consisting of all the terms on the right-hand side except the first one, is seen to approach zero as K becomes infinite. That is, in any given time interval $0 < t \leq T$, the gain constant K can be chosen so large that

$$x_n(t) = L^{-1}\hat{u}(s) + \epsilon \quad (B4)$$

and ϵ is arbitrarily small.

To relate this result to the case at hand, one must show that the closed-loop system

$$\dot{x} = Fx + BK(HFB)^{\dagger}(LH - HF)x \quad (B5)$$

has embedded within it subsystems of the type shown in sketch (b) at every integrator to which B distributes the control u . This is demonstrated by the phase variable form of the open-loop plant given below (ref. 7):

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ a_1 & a_2 & \cdot & \cdot & \cdot & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u \quad (B6)$$

In the basis which yields the above representation of the plant, the matrix H must have the form

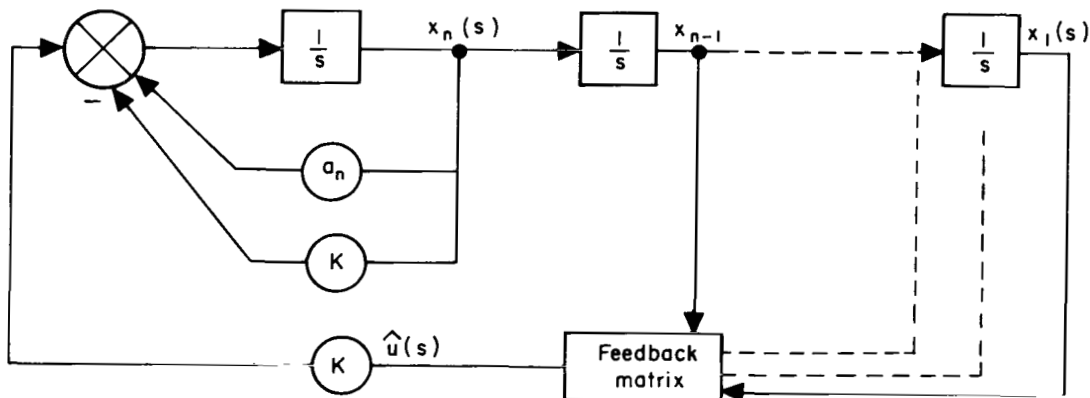
$$H = \begin{bmatrix} h_{11} & h_{12} & \cdot & \cdot & \cdot & h_{1n-1} & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & 0 \\ \cdot & & & & & & \cdot \\ & & & & & & \cdot \\ & & & & & & 0 \\ h_{l1} & & & & & h_{ln} & 0 \end{bmatrix} \quad (B7)$$

since $HB = 0$ because delta function controls are required for perfect matching. Given equations (B6) and (B7), one computes for the gain matrix

$$(HFB)^{\dagger}(LH - HF) = [k_1, k_2, \dots, -1] \quad (B8)$$

The -1 in the last entry of the gain matrix assures that the feedback gain around the n th integrator is nonzero and negative. Hence, the flow diagram around the n th integrator is of the form shown in sketch (b).

One final point must be considered. In the closed-loop case, the input, $\hat{u}(s)$, to the n th integrator itself depends on the gain K through the feedback loop, while earlier $\hat{u}(s)$ was assumed to be independent of K . It is therefore necessary in the closed-loop case to be sure that the poles and residues of $\hat{u}(s)$ converge to finite values as K approaches infinity and that the degree of the denominator of the limiting form of $\hat{u}(s)$ is greater than the degree of the numerator. Sketch (c) illustrates the problem for the



Sketch (c).- Block diagram for computing $\hat{u}(s)$.

assumed phase-variable form of the system. Straightforward computation shows $\hat{u}(s)$ has the form

$$\hat{u}(s) = \frac{f_1(s)}{s^n} x_n(s) + \frac{f_2(s)}{s^n} \quad (B9)$$

where $f_1(s)$ and $f_2(s)$ are polynomials of at most $(n-1)$ st degree. But $x_n(s)$ can also be written as a function of $\hat{u}(s)$:

$$x_n(s) = \frac{K\hat{u}(s)}{s - a_n + K} + \frac{x_n(0)}{s - a_n + K} \quad (B10)$$

After $x_n(s)$ is eliminated between equations (B9) and (B10), $\hat{u}(s)$ can be put into the form:

$$\hat{u}(s) = \frac{1}{1 - \frac{f_1(s)K}{s^n(s - a_n + K)}} \left[\frac{f_1(s)x_n(0)}{s^n(s - a_n + K)} + \frac{f_2(s)}{s^n} \right] \quad (B11)$$

The limiting form of $\hat{u}(s)$ as $K \rightarrow \infty$ therefore becomes

$$\hat{u}(s)_{\text{lim}} = \frac{f_2(s)}{s^n - f_1(s)} \quad (B12)$$

In both equation (B11), which holds for finite K , and equation (B12) the degree of the numerator is less than the degree of the denominator as required.

ALTERNATE DISCUSSION OF APPROXIMATION FOR DELTA FUNCTION CONTROLS

A disadvantage of the preceding discussion concerning approximation of delta function controls with large feedback gains is that it is difficult to generalize to vector controls or to higher order delta functions. An alternate approach that avoids this difficulty but necessitates the use of a piecewise constant control is now presented. To begin with, it is assumed that perfect matching in the implicit model-following sense can be achieved with controls containing finite, delta function, and derivative of delta function components. These three components of u (eq. (18)) are repeated here for convenience:

$$u_1 = P_1 \tilde{u}, \quad u_\delta = P_\delta \tilde{u}, \quad u_{\delta 1} = P_{\delta 1} \tilde{u} \quad (B13)$$

where \tilde{u} is defined as

$$\tilde{u} = M^\dagger (LH - HF)x \equiv (HB + HF B P_\delta + HF^2 B P_{\delta 1})^\dagger (LH - HF)x \quad (B14)$$

The next step is to divide time into equal increments Δt , which are chosen much shorter than the shortest time constant of the model and the plant. The control applied to the plant remains constant throughout each time increment and is updated only at the beginning of a new increment. If the control process is assumed to start at $t = 0$, the first control applied to the plant is chosen as

$$u(0) = \left[P_1 + \frac{1}{\Delta t} P_\delta + \frac{2}{(\Delta t)^2} P_{\delta 1} \right] M^\dagger (LH - HF)x(0) \quad (B15)$$

We note that the gain constants multiplying those components of control that require delta function and derivative of delta function are $1/\Delta t$ and $2/(\Delta t)^2$, respectively. Also, $x(0)$ is arbitrary and therefore may be such that $\dot{y}(0) \neq Ly(0)$.

This sets the stage for the crucial step of this approach, namely, the computation of the error between \dot{y} and Ly at the end of the first time increment. Using the standard expression for the time response of a linear system (ref. 6), we compute $y(\Delta t)$:

$$y(\Delta t) = H \left[e^{F\Delta t} x(0) + \int_0^{\Delta t} e^{F(\Delta t - \tau)} Bu(\tau) d\tau \right] \quad (B16)$$

where the transition matrix e^{Ft} is given in terms of the infinite series as

$$e^{Ft} = I + Ft + \frac{F^2 t^2}{2!} + \dots \quad (B17)$$

Substituting equation (B17) into (B16) and using the fact that $u(t)$ is constant within the integration interval yields

$$y(\Delta t) = H \left\{ x(0) + \Delta t Fx(0) + \frac{\Delta t^2}{2!} F^2 x(0) + \dots \right. \\ \left. + \left[I\Delta t + \frac{F\Delta t^2}{2} + \frac{F^2 \Delta t^3}{3!} + \dots \right] Bu(0) \right\} \quad (B18)$$

The term $y(\Delta t)$ is calculated to evaluate the error between $\dot{y}(\Delta t)$ and $Ly(\Delta t)$ and to show that it can be made arbitrarily small. The error, denoted by $\eta(\Delta t)$, is evaluated using equations (1), (2), (B15), and (B18):

$$\eta(\Delta t) = H \left\{ F + \Delta t F^2 + \frac{\Delta t^2 F^3}{2!} + \dots \left(F\Delta t + \frac{F^2 \Delta t^2}{2} + \frac{F^3 \Delta t^3}{3!} + \dots \right) \left[BP_1 + \frac{BP_\delta}{\Delta t} \right. \right. \\ \left. \left. + \frac{2BP_{\delta 1}}{(\Delta t)^2} \right] M^\dagger (LH - HF) + B \left[P_1 + \frac{1}{\Delta t} P_\delta + \frac{2}{(\Delta t)^2} P_{\delta 1} \right] M^\dagger (LH - HF) \right\} x(0) \\ - \left\{ LH + \Delta t LHF + \frac{\Delta t^2}{2!} LHF^2 + \dots + LH \left(I\Delta t + \frac{F\Delta t^2}{2!} + \frac{F^2 \Delta t^3}{3!} \right) \left[BP_1 \right. \right. \\ \left. \left. + \frac{BP_\delta}{\Delta t} + \frac{2BP_{\delta 1}}{(\Delta t)^2} \right] M^\dagger (LH - HF) \right\} x(0) \quad (B19)$$

Since $HBP_{\delta} = 0$, $HBP_{\delta 1} = 0$, and $HFBP_{\delta 1} = 0$, equation (B19) simplifies to

$$\eta(\Delta t) = (MM^{\dagger} - I)(LH - HF) + O(\Delta t) \quad (B20)$$

where $O(\Delta t)$ are terms that go to zero at least as fast as Δt . But the first term is identically zero by the assumption that the perfect matching condition, equation (16), is satisfied. Thus, if Δt is sufficiently small (or the gain constants arbitrarily large), the error at the end of Δt seconds can be made as small as desired for any initial condition.

Furthermore, the error can be maintained arbitrarily small for all future time if the control, equation (B15), is updated at the beginning of each new time increment by replacing $x(n\Delta t)$ with $x[(n + 1)\Delta t]$, n being the number of time increments.

Needless to say, through appropriate limiting arguments, the discussion given here for a discrete time control law can be generalized to continuous time control. The mathematical details, however, are tedious and are not elaborated here.

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